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GENERAL DEFECT RELATIONS OF HOLOMORPHIC CURVES

BY

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Dedicated to Professor Yûsaku Komatu on his 70th birthday

ABSTRACT. Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of finite lower order μ, and let $A = \{\alpha\}$ be an arbitrary finite family of holomorphic curves $\alpha\colon \mathbf{C} \to (P_n\mathbf{C})^*$ satisfying $T(r,\alpha) = o(T(r,x))$ $(r \to \infty)$. Suppose x is nondegenerate with respect to A, and A is in general position. We show the following general defect relations: (1) x has at most n deficient curves in A if $\mu = 0$. (2) $\sum_{\alpha\in A}\delta(\alpha) \le n$ if $0 < \mu \le 1/2$. (3) $\sum_{\alpha\in A}\delta(\alpha) \le [2n\mu]+1$ if $1/2 < \mu < +\infty$.

1. Introduction. Let f be a transcendental meromorphic function on the complex plane $\mathbf{C}$ and let a_j $(j = 1, 2, \ldots, q)$ be distinct meromorphic functions on $\mathbf{C}$ satisfying

$$T(r, a_j) = o(T(r, f)), \qquad r \to \infty.$$

Then the deficiency of a_j with respect to f is defined by

$$\delta(a_j, f) = 1 - \limsup_{r\to\infty} \frac{N(r, 0, f - a_j)}{T(r, f)},$$

and a_j is called a deficient function of f when $\delta(a_j, f) > 0$.

In 1929 Nevanlinna [**10**] proved that the defect relation

$$\sum_{j=1}^{q} \delta(a_j, f) \le 2 \tag{1.1}$$

is valid for $q = 3$ and asked if (1.1) is true for all positive integers q. In 1939 Dufresnoy [**4**] showed that $\sum_{j=1}^{q}\delta(p_j, f) \le 2 + d$ for distinct polynomials p_j of degree $\le d$. In 1964 Chuang [**3**] proved (1.1) when $\delta(\infty, f) = 1$. Recently, Yang [**17**] gave the following:

THEOREM A. *Let μ be the lower order of f.*
(1) *If $\mu = 0$, f has at most one deficient function* (*cf. Mori* [**8**, *Theorem* 1]).
(2) *If $\mu > 0$ then*

$$\sum_{j=1}^{q}\delta(a_j, f)\begin{cases} \le 1 & (0 < \mu \le 1/2,\ q = 1), \\ < 1 - \cos\pi\mu & (0 < \mu \le 1/2,\ q \ge 2), \\ \le 2 - \sin\pi\mu & (1/2 < \mu \le 1), \\ \le \min\{[2\mu] + 1, (\sqrt{2}/2)\pi\mu\} & (\mu > 1). \end{cases}$$

On the other hand, Mori [**9**] regarded $\delta(a_j, f)$ as a deficiency of the moving divisor and extended the above result of Nevanlinna to the case of holomorphic

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mappings of $\mathbf{C}^n$ into $P_m\mathbf{C}$. Further, under some conditions, Shiffman [**12**], Mori [**9**], and Stoll [**13**] obtained general defect relations for moving divisors with respect to meromorphic functions on $\mathbf{C}^n$, holomorphic mappings of $\mathbf{C}^n$ into $P_m\mathbf{C}$, and holomorphic mappings of M into $P_m\mathbf{C}$, respectively, where M is a parabolic complex manifold.

The purpose of this paper is to extend Theorem A to the case of holomorphic curves. We assume the reader is familiar with Nevanlinna theory of meromorphic functions and holomorphic curves (cf. [**10**, **15**, **16**]).

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2. Notations and main results. We denote complex projective n-space by $P_n\mathbf{C}$ and dual complex projective n-space by $(P_n\mathbf{C})^*$. Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve and $\tilde{x} = (x_0, x_1, \dots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ its reduced representation. We define the characteristic (order) function $T(r, x)$ of x by

$$T(r,x) = \frac{1}{2\pi}\int_0^{2\pi} \log|\tilde{x}(re^{i\theta})|\,d\theta - \log|\tilde{x}(0)|, \tag{2.1}$$

where $|\tilde{x}(z)| = (\sum_{j=0}^n |x_j(z)|^2)^{1/2}$.

Let $\alpha\colon \mathbf{C} \to (P_n\mathbf{C})^*$ be a holomorphic curve and $\tilde{\alpha} = (\alpha_0, \alpha_1, \dots, \alpha_n)\colon \mathbf{C} \to (\mathbf{C}^{n+1})^* - \{0\}$ its reduced representation. We denote by $N(r,\alpha) \equiv N(r,\alpha;x)$ the counting function of zeros of the entire function

$$F(z) \equiv \langle \tilde{x}(z), \tilde{\alpha}(z)\rangle = \sum_{j=0}^n \alpha_j(z)x_j(z) \not\equiv 0,$$

that is, $N(r,\alpha) = N(r,0,F)$. We define the deficiency $\delta(\alpha) \equiv \delta(\alpha, x)$ of α with respect to x by

$$\delta(\alpha) = 1 - \limsup_{r\to\infty} \frac{N(r,\alpha)}{T(r,\alpha) + T(r,x)}. \tag{2.2}$$

Since $|F(z)| \le |\tilde{\alpha}|\,|\tilde{x}|$, (2.1) and Jensen's formula imply $0 \le \delta(\alpha) \le 1$. If $\delta(\alpha) > 0$ then we say that α is a deficient curve. We remark that if α satisfies

$$T(r,\alpha) = o(T(r,x)), \qquad r \to \infty, \tag{2.3}$$

then (2.2) is reduced to

$$\delta(\alpha) = 1 - \limsup_{r\to\infty} \frac{T(r,\alpha)}{T(r,x)}. \tag{2.4}$$

Now let $\alpha^{(k)}\colon \mathbf{C} \to (P_n\mathbf{C})^*$ $(k = 1, 2, \dots, q;\ q > n+1)$ be holomorphic curves and $\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \dots, \alpha_n^{(k)})\colon \mathbf{C} \to (\mathbf{C}^{n+1})^* - \{0\}$ their reduced representation. If $\det\{(\alpha^{(k_i)}(z))_{1\le i\le n+1,\ 0\le j\le n}\} \not\equiv 0$ for any $\{k_1, k_2, \dots, k_{n+1}\} \subset \{1, 2, \dots, q\}$, then we say that $\{\alpha^{(k)}\}_{k=1}^q$ is in general position. A curve $x\colon \mathbf{C} \to P_n\mathbf{C}$ is called nondegenerate with respect to $\{\alpha^{(k)}\}$ if

$$F_k(z) = \sum_{j=0}^n \alpha_j^{(k)}(z)x_j(z) \not\equiv 0 \quad \text{for all } k.$$

Using Yang's idea [**17**], we prove the following defect relations:

THEOREM 1. *Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of lower order μ and $A = \{\alpha\}$ an arbitrary finite family of holomorphic curves $\alpha\colon \mathbf{C} \to (P_n\mathbf{C})^*$ satisfying*

$$T(r,\alpha) = o(T(r,x)), \qquad r \to \infty. \tag{2.3}$$

Suppose x is nondegenerate with respect to A, and A is in general position. Then:

(I) *If $\mu = 0$ there are at most n deficient curves in A.*

(II) *Assume $0 < \mu \le 1/2$ and put $B = \{\alpha \in A; \delta(\alpha) > 1 - \cos\pi\mu\}$ and $C = \{\alpha \in A;\ \delta(\alpha) = 1 - \cos\pi\mu\}$. If there are n curves belonging to $B \cup C$, then all the remaining deficiencies are equal to zero. If $\#B = p < n$, then*

$$\sum_{\alpha\in A-B} \delta(\alpha) \le (n-p)(1-\cos\pi\mu),$$

where equality holds if and only if $\#C = n-p$ and $\delta(\alpha) = 0$ for all $\alpha \in A-(B\cup C)$.

(III) *If $1/2 < \mu < +\infty$ we have*

$$\sum_{\alpha\in A} \delta(\alpha) \le [2n\mu] + 1 - \cos(2n\mu - [2n\mu])(\pi/2).$$

As an immediate consequence of (I) and (II) we have

COROLLARY. *If $0 \le \mu \le 1/2$, then $\sum_{\alpha\in A} \delta(\alpha) \le n$.*

3. Spread relation. Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of lower order μ $(0 < \mu < +\infty)$ and $\tilde{x} = (x_0, x_1, \dots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ its reduced representation. Suppose that a holomorphic curve $\alpha\colon \mathbf{C} \to (P_n\mathbf{C})^*$ and its reduced representation $\tilde{\alpha}\colon \mathbf{C} \to (\mathbf{C}^{n+1})^* - \{0\}$ satisfy

$$T(r,\alpha) = o(T(r,x)), \qquad r \to \infty, \tag{3.1}$$

and $\langle \tilde{x}(z), \tilde{\alpha}(z)\rangle \not\equiv 0$. We put

$$\|x(z), \alpha(z)\| = |\langle \tilde{x}(z), \tilde{\alpha}(z)\rangle| / (|\tilde{x}(z)|\,|\tilde{\alpha}(z)|).$$

A positive, increasing, unbounded sequence $\{r_m\}$ is called a sequence of Pólya peaks of order μ of $T(r,x)$ if it is possible to find positive sequences $\{r'_m\}$, $\{r''_m\}$, and $\{\varepsilon_m\}$ such that, as $m \to \infty$, then $r'_m \to \infty$, $r_m/r'_m \to \infty$, $r''_m/r_m \to \infty$, $\varepsilon_m \to 0$, and

$$T(t,x)/T(r_m,x) \le (t/r_m)^\mu (1+\varepsilon_m) \qquad (r'_m < t < r''_m). \tag{3.2}$$

We now define the spread $\sigma(\alpha) \equiv \sigma(\alpha, x)$ of α with respect to x. Let $\{r_m\}$ be a sequence of Pólya peaks of order μ of $T(r,x)$ and $\Lambda(r)$ a positive function satisfying

$$\Lambda(r) = o(T(r,x)), \qquad r \to \infty. \tag{3.3}$$

Put

$$E_\Lambda(r,\alpha) \equiv E_\Lambda(r,\alpha;x) = \{\theta; \log\|x(re^{i\theta}), \alpha(re^{i\theta})\| < -\Lambda(r)\} \subset (-\pi, \pi]$$

and let

$$\sigma_\Lambda(\alpha) = \liminf_{m\to\infty} \operatorname{meas} E_\Lambda(r_m, \alpha).$$

Then we define

$$\sigma(\alpha) = \inf_\Lambda \sigma_\Lambda(\alpha),$$

where the infimum is taken over all functions $\Lambda(r)$ satisfying (3.3).

We obtain the following spread relation, which is a generalization of Baernstein [**1**]—Yang [**17**] and the author [**11**]—Krytov [**7**].

THEOREM 2 (SPREAD RELATION). *Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of lower order μ $(0 < \mu < +\infty)$. Then*

$$\sigma(\alpha) \geq \min\{2\pi, (4/\mu)\sin^{-1}(\delta(\alpha)/2)^{1/2}\}$$

for every holomorphic curve $\alpha\colon \mathbf{C} \to (P_n\mathbf{C})^*$ satisfying (3.1) *and* $\|x(z), \alpha(z)\| \not\equiv 0$.

PROOF. We may assume that the reduced representations $\tilde{x}$ and $\tilde{\alpha}$ of x and α, respectively, satisfy

$$|\tilde{x}(0)| = 1 \quad \text{and} \quad |\tilde{\alpha}(0)| = 1. \tag{3.4}$$

We have

$$E_\Lambda(r, \alpha) = \{\theta; \log|\tilde{x}(re^{i\theta})| + \log|\tilde{\alpha}(re^{i\theta})| - \log|F(re^{i\theta})| > \Lambda(r)\},$$

where

$$F(z) = \langle \tilde{x}(z), \tilde{\alpha}(z)\rangle = \sum_{j=0}^{n} \alpha_j(z)x_j(z).$$

Define the entire function $h(z)$ by

$$h(z) = cz^{-k}F(z),$$

where k is a nonnegative integer, c is a nonzero constant, and

$$h(0) = 1. \tag{3.5}$$

Then

$$N(r, \alpha) = N(r, 0, F) = N(r, 0, h) + k\log r. \tag{3.6}$$

We put

$$\begin{gathered} G(z) = \log|\tilde{x}(z)| + \log|\tilde{\alpha}(z)| - \log|h(z)|, \\ \Lambda_1(r) = \Lambda(r) + k\log r - \log|c|, \end{gathered} \tag{3.7}$$

and

$$E_{\Lambda_1}(r, G) = \{\theta; G(re^{i\theta}) > \Lambda_1(r)\}.$$

Then we deduce that

$$E_{\Lambda_1}(r, G) = E_\Lambda(r, \alpha), \tag{3.8}$$

and

$$\Lambda_1(r) = o(T(r, x)) \qquad (r \to \infty). \tag{3.9}$$

We next write $T(r) = T(r, x) + T(r, \alpha)$. Then it follows from (3.1) that

$$T(r) \sim T(r, x), \qquad r \to \infty, \tag{3.10}$$

that is, $T(r)/T(r,x) \to 1$ as $r \to 1$. Hence, $T(r)$ and $T(r, x)$ have the same sequence of Pólya peaks, and, combining (3.6) and (3.10) with the definition of deficiency $\delta(\alpha)$, we have

$$\delta(\alpha) = 1 - \limsup_{r\to\infty} \frac{N(r, 0, h)}{T(r)}. \tag{3.11}$$

Therefore, in order to prove Theorem 2, from (3.8)–(3.11) it is sufficient to prove that

$$\liminf_{r\to\infty} \operatorname{meas} E_{\Lambda_1}(r_m, G) \geq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha)}{2}\right)^{1/2}\right\}. \tag{3.12}$$

Following Baernstein [**1**] we now define

$$T^*(z) = \sup_E \frac{1}{2\pi}\int_E G(re^{i\omega})\,d\omega + N(r,0,h) \qquad (z = re^{i\theta},\ 0 < r < +\infty,\ 0 \leq \theta \leq \pi),$$

where the supremum is taken over all measurable sets $E \subset (-\pi, \pi]$ whose measure equals 2θ. Further, we define

$$T^*(0) = 0. \tag{3.13}$$

Then since $\log|\tilde{x}(z)| + \log|\tilde{\alpha}(z)|$ and $\log|h(z)|$ are subharmonic, from (3.4), (3.5), (3.7), (3.13), and Theorem A′ in Baernstein [**2**], it follows that $T^*(z)$ is subharmonic in $\{z;\ \operatorname{Im} z > 0\}$ and continuous on $\{z;\ \operatorname{Im} z \geq 0\}$. By definition of $T^*(z)$,

$$T^*(re^{i\pi}) = T^*(-r) = T(r), \quad T^*(r) = N(r,0,h) \equiv N(r). \tag{3.14}$$

Since $|h(z)| \leq |c|\,|z|^{-k}|\tilde{\alpha}(z)|\,|\tilde{x}(z)|$, then $G(z) + \log|c| - k\log|z| \geq 0$ for all z. Hence, we deduce that

$$\begin{aligned} T^*(re^{i\theta}) &\leq T^*(re^{i\pi}) + ((\pi-\theta)/\pi)\{\log|c| - k\log r\} \\ &< T(r) + K \qquad (r > 1), \end{aligned} \tag{3.15}$$

where K is a positive constant such that $\log^+|c| < K$.

Now we can apply the arguments of Baernstein [**1**, pp. 430–433] and the author [**11**, pp. 364–365]. For the sake of clarification we sketch them here.

If $\delta(\alpha) = 0$ there is nothing to prove, so we may assume $\delta(\alpha) > 0$. Put

$$\gamma = (1/2\pi)\min\{2\pi, (4/\mu)\sin^{-1}(\delta(\alpha)/2)^{1/2}\}.$$

Then

$$0 < \gamma \leq 1, \quad 0 < \gamma\mu \leq 1/2 \quad \text{and} \quad 1 - \delta(\alpha) \leq \cos\pi\gamma\mu. \tag{3.16}$$

Define

$$v(z) = T^*(z^\gamma) \qquad (z = re^{i\theta},\ 0 < r < \infty,\ 0 \leq \theta \leq \pi).$$

Then $v(z)$ is subharmonic in $\{z;\ \operatorname{Im} z > 0\}$ and continuous in $\{z;\ \operatorname{Im} z \geq 0\}$. Hence, when $re^{i\theta} \in D_R = \{z = re^{i\theta};\ 0 < r < R,\ 0 < \theta < \pi\}$,

$$v(re^{i\theta}) \leq \int_{-R}^{R} v(t)A(t,r,\theta,R)\,dt + \int_0^\pi v(Re^{i\psi})B(\psi,r,\theta,R)\,d\psi, \tag{3.17}$$

where

$$A(t,r,\theta,R) = \frac{1}{\pi}\frac{r\sin\theta}{t^2 - 2tr\cos\theta + r^2} - \frac{1}{\pi}\frac{R^2 r\sin\theta}{R^4 - 2rtR^2\cos\theta + r^2t^2},$$

$$B(\psi,r,\theta,R) = \frac{2Rr\sin\theta}{\pi}\frac{(R^2 - r^2)\sin\psi}{|R^2e^{2i\psi} - 2rRe^{i\psi}\cos\theta + r^2|^2}.$$

It follows from (3.14) and (3.15) that

$$v(t) = T^*(t^\gamma) = N(t^\gamma), \quad v(-t) \leq T(r) + K, \quad v(Re^{i\psi}) \leq T(r) + K. \tag{3.18}$$

We deduce that

$$(3.19)\quad 0 < A(t,r,\theta,R) \leq \frac{1}{\pi}\frac{r\sin\theta}{t^2-2tr\cos\theta+r^2} \equiv P(t,r,\pi-\theta) \qquad (0<r<R,\ 0<\theta<\pi),$$

$$(3.20)\quad 0 < B(\psi,r,\theta,R) \leq 32r/\pi R \qquad (0<\theta<\pi,\ 0<\psi<\pi,\ 0<r<R/2),$$

and

$$(3.21)\quad \int_0^\infty P(t,r,\theta)\,d\theta = \frac{\theta}{\pi} < 1 \qquad (0<\theta<\pi,\ r>0).$$

Using (3.18)–(3.21) in (3.17) we obtain

$$(3.22)\quad v(re^{i\theta}) \leq \int_0^R N(t^\gamma)P(t,r,\pi-\theta)\,dt + \int_0^R T(t^\gamma)P(t,r,\theta)\,dt + 32(r/R)(T(R^\gamma)+K)+K \qquad (0<\theta<\pi,\ 0<r<R/2).$$

Let $\{r_m\}$ be any sequence of Pólya peaks of order μ of $T(r,x)$ and put $s_m = r_m^{1/\gamma}$. $\{r_m\}$ is also a sequence of Pólya peaks of order μ of $T(r)$. Hence, it follows from the discussion of Baernstein [**1**, pp. 332–333] that

$$(3.23)\quad v(s_m e^{i\theta}) \leq T(r_m)\{\cos(\pi-\theta)\gamma\mu + \alpha_m\} \qquad (m=1,2,\ldots;\ 0<\theta<\pi),$$

where $\{\alpha_m\}$ is a sequence tending to zero. Let

$$\sigma_m = \operatorname{meas} E_{\Lambda_1}(r_m, G) \equiv \operatorname{meas} E(r_m).$$

Then (3.12) is equivalent to the inequality

$$(3.24)\quad \liminf_{m\to\infty} \sigma_m \geq 2\pi\gamma.$$

It follows from (3.4) and (3.5) that

$$\begin{aligned} T(r_m) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\log|\tilde{x}(r_m e^{i\omega})| + \log|\tilde{\alpha}(r_m e^{i\omega})|)\,d\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} G(r_m e^{i\omega})\,d\omega + N(r_m) \\ &\leq \frac{1}{2\pi}\int_{E(r_m)} G(r_m e^{i\omega})\,d\omega + N(r_m) + \Lambda_1(r_m) \\ &\leq T^*(r_m e^{i\sigma_m/2}) + \Lambda_1(r_m). \end{aligned}$$

Dividing by $T(r_m)$ and remembering (3.15), we find that

$$(3.25)\quad \lim_{m\to\infty} \frac{T^*(r_m e^{i\sigma_m/2})}{T(r_m)} = 1.$$

Let $M = \{m; \sigma_m < 2\pi\gamma\}$. If M is a finite set, then (3.24) holds and we have finished, so we assume that M is infinite. The point $(r_m e^{i\sigma_m/2})^{1/\gamma} = s_m e^{i\sigma_m/2\gamma}$ belongs to the domain of $v(z)$, i.e. the upper half-plane, if and only if $m \in M$, in which case we have

$$T^*(r_m e^{i\sigma_m/2}) = v(s_m e^{i\sigma_m/2\gamma}) \qquad (m \in M).$$

Using this in (3.25), comparing with (3.23), and remembering (3.16), we deduce that $\lim_{m\to\infty,\ m\in M} \sigma_m/2\gamma = \pi$, which shows that (3.24) holds in this case also.

Thus the proof of Theorem 2 is complete.

4. Lemmas. In order to prove Theorem 1 we need several lemmas.

LEMMA 1. *Let* $x\colon \mathbf{C} \to P_n\mathbf{C}$ *be a holomorphic curve,* $\alpha^{(k)}\colon \mathbf{C} \to (P_n\mathbf{C})^*$ $(k = 1, 2, \ldots, n+1)$ *holomorphic curves in general position, and* $\Lambda(r)$ *a positive function satisfying* (3.2). *Suppose* x *is nondegenerate with respect to* $\{\alpha^{(k)}\}_{k=1}^{n+1}$. *Then*

$$\operatorname{meas}\left\{\bigcap_{k=1}^{n+1} E_\Lambda(r, \alpha^{(k)}, x)\right\} \leq \frac{2\pi}{\Lambda(r)}\left\{\sum_{k=1}^{n+1} T(r, \alpha^{(k)}) + K\right\},$$

where K *is a constant independent of* Λ.

PROOF. Let $\tilde{x} = (x_0, x_1, \ldots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ and

$$\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_n^{(k)})\colon \mathbf{C} \to (\mathbf{C}^{n+1})^* - \{0\}$$

be reduced representations of x and $\alpha^{(k)}$, respectively. By the linear equations

$$\sum_{j=0}^{n} \alpha_j^{(k)}(z)x_j(z) = F_k(z), \qquad k = 1, 2, \ldots, n+1, \tag{4.1}$$

we have $x_j = D_j/D$, where D is the coefficient determinant of (4.1) and

$$D_j = \sum_{k=1}^{n+1}(-1)^{j+k+1}D_{kj}F_k,$$

where D_{kj} is the minor of D obtained by omitting the kth row and $(j+1)$th column from D. The Hadamard inequality yields

$$|D_{kj}| \leq \prod_{l=1,\neq k}^{n+1} |\tilde{\alpha}^{(l)}|,$$

so

$$|x_j| \leq \frac{1}{|D|}\sum_{k=1}^{n+1}\left(\prod_{l=1,\neq k}^{n+1} |\tilde{\alpha}^{(l)}|\right)|F_k|.$$

Hence,

$$\log|\tilde{x}| \leq \log\left\{\sum_{k=1}^{n+1}\left(\prod_{l=1,\neq k}^{n+1}|\tilde{\alpha}^{(l)}|\right)|F_k|\right\} - \log|D| + \tfrac{1}{2}\log(n+1). \tag{4.2}$$

For $z = re^{i\theta}$, with $\theta \in E(r) \equiv \bigcap_{k=1}^{n+1} E_\Lambda(r, \alpha^{(k)})$, we have

$$|F_k(z)|/(|\tilde{x}(z)|\,|\tilde{\alpha}^{(k)}(z)|) < \exp(-\Lambda(r)), \qquad k = 1, 2, \ldots, n+1,$$

so

$$\left\{\sum_{k=1}^{n+1}\left(\prod_{l=1,\neq k}^{n+1}|\tilde{\alpha}^{(l)}|\right)|F_k|\right\}\Bigg/\left(|\tilde{x}|\prod_{k=1}^{n+1}|\alpha^{(k)}|\right) < (n+1)\exp(-\Lambda(r)).$$

Hence, combining this with (4.2) we obtain

$$\sum_{k=1}^{n+1}\log|\tilde{\alpha}^{(k)}(z)| - \log|D(z)| > \Lambda(r) - \frac{3}{2}\log(n+1) \text{ for } z = re^{i\theta},\ \theta \in E(r). \tag{4.3}$$

The Hadamard inequality implies $|D(z)| \leq \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}(z)|$, so

$$\sum_{k=1}^{n+1} \log|\tilde{\alpha}^{(k)}(z)| - \log|D(z)| \geq 0 \tag{4.4}$$

for all z. Hence, it follows from (4.3) and (4.4) that

$$\begin{aligned}
&\frac{1}{2\pi}(\Lambda(r) - \frac{3}{2}\log(n+1)) \operatorname{meas} E(r) \\
&\quad \leq \frac{1}{2\pi}\int_{E(r)} \left(\sum_{k=1}^{n+1} \log|\tilde{\alpha}^{(k)}(re^{i\theta})| - \log|D(re^{i\theta})|\right) d\theta \\
&\quad \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \left(\sum_{k=1}^{n+1} \log|\tilde{\alpha}^{(k)}(re^{i\theta})| - \log|D(re^{i\theta})|\right) d\theta \\
&\quad = \sum_{k=1}^{n+1}(T(r,\alpha^{(k)}) + \log|\tilde{\alpha}^{(k)}(0)|) - (N(r,0,D) + \log|c_\lambda|),
\end{aligned}$$

where

$$D(z) = c_\lambda z^\lambda + c_{\lambda+1}z^{\lambda+1} + \cdots \qquad (c_\lambda \neq 0).$$

Hence, choosing a constant K such that

$$\sum_{k=1}^{n+1} \log|\tilde{\alpha}^{(k)}(0)| - \log|c_\lambda| + \frac{3}{2}\log(n+1) < K,$$

we obtain

$$\operatorname{meas} E(r) \leq \frac{2\pi}{\Lambda(r)}\left\{\sum_{k=1}^{n+1} T(r,\alpha^{(k)}) + K\right\},$$

which proves Lemma 1.

Now let $A = \{\alpha^{(k)}\}_{k=1}^{q}$ $(q \geq n+1)$ be a finite family of holomorphic curves in general position which satisfy

$$T(r,\alpha^{(k)}) = o(T(r,x)), \qquad r \to \infty. \tag{4.5}$$

Further suppose that x is nondegenerate with respect to A. We define $I_\Lambda(r,\alpha^{(k)}) \equiv I_\Lambda(r,\alpha^{(k)},A)$ by

$$\begin{aligned}
I_\Lambda(r,\alpha^{(k)}) &= E_\Lambda(r,\alpha^{(k)}) \\
&\quad - \bigcup_{(l_1,\ldots,l_n)} \left\{E_\Lambda(r,\alpha^{(k)}) \cap \left(\bigcap_{j=1}^{n} E_\Lambda(r,\alpha^{(l_j)})\right)\right\},
\end{aligned} \tag{4.6}$$

where the union is taken over all possible n-tuples $(l_1,\ldots,l_n) \subset (1,\ldots,k-1,k+1,\ldots,q)$, and

$$\sigma'_\Lambda(\alpha^{(k)}) = \liminf_{m\to\infty} \operatorname{meas} I_\Lambda(r_m,\alpha^{(k)}). \tag{4.7}$$

Then we can deduce

LEMMA 2. *For any* $(k_1, \dots, k_{n+1}) \subset (1, 2, \dots, q)$ *we have*

$$\bigcap_{j=1}^{n+1} I_\Lambda(r, \alpha^{(k_j)}) = \emptyset.$$

We now prove

LEMMA 3. *There is a sequence of positive functions* $\Lambda_\nu(r)$ *satisfying* (3.3) *such that*

$$\liminf_{\nu\to\infty} \sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha^{(k)})}{2}\right)^{1/2}\right\}.$$

PROOF. We can choose a constant K such that Lemma 1 is valid for all $(n+1)$-tuples $(k_1, \dots, k_{n+1}) \subset (1, \dots, q)$. Hence, it follows that

$$\begin{aligned}
&\sum_{(l_1,\dots,l_n)} \operatorname{meas}\left\{E_\Lambda(r, \alpha^{(k)}) \cap \left(\bigcap_{j=1}^{n} E_\Lambda(r, \alpha^{(l_j)})\right)\right\} \\
&\leq \sum_{(l_1,\dots,l_n)} \frac{2\pi}{\Lambda(r)}\left\{T(r, \alpha^{(k)}) + \sum_{j=1}^{n} T(r, \alpha^{(l_j)}) + K\right\} \\
&\leq \frac{2\pi}{\Lambda(r)}\binom{q-1}{n}\left(\sum_{l=1}^{q} T(r, \alpha^{(l)}) + K\right).
\end{aligned} \tag{4.8}$$

Choose a sequence of positive numbers ε_ν such that $\lim_{\nu\to\infty}\varepsilon_\nu = 0$. Put

$$\Lambda_\nu(r) = \frac{2\pi}{\varepsilon_\nu}\binom{q-1}{n}\left(\sum_{l=1}^{q} T(r, \alpha^{(l)}) + K\right). \tag{4.9}$$

Then it follows from (4.5) that for every fixed ν, $\Lambda_\nu(r)$ satisfies (3.3). From the definition of $\sigma_\Lambda(\alpha)$ and (4.6)–(4.9) we deduce that, for every fixed ν,

$$\begin{aligned}
\operatorname{meas} I_{\Lambda_\nu}(r_m, \alpha^{(k)}) \geq\ &\operatorname{meas} E_{\Lambda_\nu}(r_m, \alpha^{(k)}) \\
&- \sum_{(l_1,\dots,l_n)} \operatorname{meas}\left\{E_{\Lambda_\nu}(r_m, \alpha^{(k)}) \cap \left(\bigcap_{j=1}^{n} E_{\Lambda_\nu}(r_m, \alpha^{(l_j)})\right)\right\} \\
\geq\ &\operatorname{meas} E_{\Lambda_\nu}(r_m, \alpha^{(k)}) - \varepsilon_\nu,
\end{aligned}$$

so

$$\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \sigma_{\Lambda_\nu}(\alpha^{(k)}) - \varepsilon_\nu.$$

Therefore, our spread relation implies

$$\sigma'_{\Lambda_\nu}(\alpha^{(k)}) \geq \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha^{(k)})}{2}\right)^{1/2}\right\} - \varepsilon_\nu$$

for every fixed ν. Here let $\nu \to \infty$. Then we obtain Lemma 3.

We deduce the following obvious

LEMMA 4. *Let $E_1, \ldots, E_q$ be measurable subsets of $\{|z| = 1\}$. Define $E = E_1 \cup \cdots \cup E_q$. Take an integer p with $1 \leq p \leq q$. Assume that for any selection of integers $1 \leq i_1 < i_2 < \cdots < i_p \leq q$, we have $E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_p} = \emptyset$. Then*

$$\sum_{j=1}^{q} \operatorname{meas} E_j \leq (p-1) \operatorname{meas} E.$$

We finally prove

LEMMA 5. *Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of lower order μ $(0 < \mu < +\infty)$ and $\alpha^{(k)}\colon \mathbf{C} \to (P_n\mathbf{C})^*$, $k = 1, \ldots, q$ $(q \geq n+1)$ holomorphic curves satisfying* (4.5). *Suppose that x is nondegenerate with respect to $\{\alpha^{(k)}\}$, and $\{\alpha^{(k)}\}$ is in general position. Then*

$$\sum_{k=1}^{q} \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha^{(k)})}{2}\right)^{1/2}\right\} \leq 2n\pi.$$

PROOF. Lemmas 2 and 4 imply

$$\sum_{k=1}^{q} \operatorname{meas} I_\Lambda(r, \alpha^{(k)}) \leq 2n\pi,$$

and, consequently,

$$\sum_{k=1}^{q} \sigma'_\Lambda(\alpha^{(k)}) \leq 2n\pi.$$

Hence, we deduce from Lemma 3 that

$$\sum_{k=1}^{q} \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha^{(k)})}{2}\right)^{1/2}\right\} \leq 2n\pi,$$

which proves Lemma 5.

5. Proof of Theorem 1(I). Theorem 1(I) is an immediate consequence of the Corollary to Theorem 3, which is a generalization of results of Edrei-Fuchs [**6**], Toda [**14**], and Mori [**8**] (cf. Yang [**17**]).

THEOREM 3. *Let $x\colon \mathbf{C} \to P_n\mathbf{C}$ be a holomorphic curve of lower order μ and $\alpha^{(k)}\colon \mathbf{C} \to (P_n\mathbf{C})^*$ $(k = 1, \ldots, n+1)$ holomorphic curves satisfying $T(r, \alpha^{(k)}) = o(T(r,x))$ $(r \to \infty)$. Assume that x is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+1}$, $\{\alpha^{(k)}\}_{k=1}^{n+1}$ is in general position, and $\alpha^{(k)}$ $(k = 1, \ldots, n+1)$ are deficient curves with respect to x. If $\gamma = \max_{1 \leq k \leq n+1}\{1 - \delta(\alpha^{(k)})\}$, then*

$$\mu \geq \begin{cases} \frac{\log(1/\gamma(2-\gamma))}{\log(1+4/\gamma(1-\gamma))} & (\gamma \neq 0), \\ 1 & (\gamma = 0). \end{cases}$$

COROLLARY. *Holomorphic curves with more than n deficient curves in general position have a positive lower order.*

PROOF OF THEOREM 3. Let $\tilde{x} = (x_0, x_1, \ldots, x_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ and $\tilde{\alpha}^{(k)} = (\alpha_0^{(k)}, \alpha_1^{(k)}, \ldots, \alpha_n^{(k)})\colon \mathbf{C} \to (\mathbf{C}^{n+1})^* - \{0\}$ be reduced representations of x

and $\alpha^{(k)}$, respectively. Then we can choose a nonzero complex number β such that a reduced representation of a constant curve $\alpha^{(n+2)}\colon \mathbf{C} \to (P_n\mathbf{C})^*$ is $\tilde{\alpha}^{(n+2)} = (1, \beta, \beta^2, \ldots, \beta^n)$, $\{\alpha^{(k)}\}_{k=1}^{n+2}$ is in general position, and x is nondegenerate with respect to $\{\alpha^{(k)}\}_{k=1}^{n+2}$.

We now consider the holomorphic curve $y\colon \mathbf{C} \to P_n\mathbf{C}$ which Mori [**9**] constructed from x and $\{\alpha^{(k)}\}_{k=1}^{n+2}$. We put

$$F_k(z) = \sum_{j=0}^{n} \alpha_j^{(k)}(z) x_j(z).$$

Let $G = \det\{\alpha_j^{(k)}\}_{0\le j\le n,\, 1\le k\le n+1}$, G_k be the determinant of the matrix replaced the kth column vector ${}^t(\alpha_0^{(k)}, \ldots, \alpha_n^{(k)})$ of the matrix $\{\alpha_j^{(k)}\}_{0\le j\le n,\, 1\le k\le n+1}$ by the column vector ${}^t(\alpha_0^{(n+2)}, \ldots, \alpha_n^{(n+2)})$, and $g_k = G_k/G$. We denote by $h(z)$ a common factor among $g_1(z)F_1(z), \ldots, g_n(z)F_n(z)$ and $g_{n+1}(z)F_{n+1}(z)$ up to non-vanishing entire functions such that $y_{k-1}(z) = g_k(z)F_k(z)/h(z)$ $(k = 1, \ldots, n+1)$ are entire. Then, by the reasoning of Mori [**9**, proof[1] of Theorem 3.4], the holomorphic curve $y\colon \mathbf{C} \to P_n\mathbf{C}$ induced by $\tilde{y} = (y_0, y_1, \ldots, y_n)\colon \mathbf{C} \to \mathbf{C}^{n+1} - \{0\}$ has the properties

$$T(r,x) \sim T(r,y) \qquad (r \to \infty), \tag{5.2}$$

and, for $k = 1, 2, \ldots, n+1$,

$$N(r,0,y_{k-1}) = N(r,0,F_k) + o(T(r,x)) \qquad (r \to \infty). \tag{5.3}$$

Now we prove (5.2) and (5.3). Hadamard's inequality implies

$$|G_k| \le B \prod_{j=1,\, \ne k}^{n+1} |\tilde{\alpha}^{(j)}|, \tag{5.4}$$

where $B = |\tilde{\alpha}^{(n+2)}|$ is a constant. Using $|F_k| \le |\tilde{x}|\,|\tilde{\alpha}^{(k)}|$, together with (5.4), we have

$$|y_{k-1}| = \left|\frac{G_k F_k}{hG}\right| \le B|\tilde{x}| \prod_{j=1}^{n+1} \frac{|\tilde{\alpha}^{(j)}|}{|hG|},$$

so

$$|\tilde{y}| \le (n+1)^{1/2} B|\tilde{x}| \prod_{j=1}^{n+1} \frac{|\tilde{\alpha}^{(j)}|}{|hG|}.$$

Hence, we have, with a suitable constant K_1,

$$T(r,y) \le T(r,x) + \sum_{j=1}^{n+1} T(r,\alpha^{(j)}) - N(r,0,G) - N(r,0,h) + N(r,\infty,h) + K_1.$$

Since $N(r,\infty,h) \le N(r,0,G)$, we obtain

$$T(r,y) \le T(r,x) + \sum_{j=1}^{n+1} T(r,\alpha^{(j)}) + K_1. \tag{5.5}$$

[1]There is a mistake in Mori's proof of his Theorem 3.4 [**9**], but he provides a correct proof in the "Correction to [**9**]". Unfortunately, his correction will not appear in the published paper, so we prove (5.2) and (5.3) along the lines of his correct proof.

On the other hand, it follows from

$$G_k \sum_{j=0}^{n} \alpha_j^{(k)} x_j = Ghy_{k-1} \qquad (k = 1, \dots, n+1)$$

that $x_j = GhH_j/H$, where $H = \det\{G_k\alpha_j^{(k)}\}_{1\le k\le n+1,\, 0\le j\le n}$ and H_j is the determinant of the matrix obtained by replacing the jth column vector

$${}^{\mathrm{t}}(G_1\alpha_j^{(1)}, \dots, G_{n+1}\alpha_j^{(n+1)})$$

of the matrix $\{G_k\alpha_j^{(k)}\}_{1\le k\le n+1,\, 0\le j\le n}$ by the column vector ${}^{\mathrm{t}}(y_0, \dots, y_n)$. We put $\tilde{\eta}_k = {}^{\mathrm{t}}(G_1\alpha_k^{(1)}, \dots, G_{n+1}\alpha_k^{(n+1)})$. Then using Hadamard's inequality together with (5.4), we have

$$|H_j| \le |\tilde{y}| \prod_{k=0,\, \ne j}^{n} |\tilde{\eta}_k|,$$

and

$$|\tilde{\eta}_k|^2 = \sum_{l=1}^{n+1} |G_l\alpha_k^{(l)}|^2 \le \sum_{l=1}^{n+1} \left(B \prod_{j=1,\, \ne l}^{n+1} |\tilde{\alpha}^{(j)}| \right)^2 |\tilde{\alpha}^{(l)}|^2 = (n+1)B^2 \prod_{j=1}^{n+1} |\tilde{\alpha}^{(j)}|^2,$$

so

$$|H_j| \le |\tilde{y}| \left\{ (n+1)^{1/2} B \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}| \right\}^n .$$

By Hadamard's inequality we also have

$$|G| \le \prod_{k=1}^{n+1} |\tilde{\alpha}^{(k)}|. \tag{5.6}$$

Hence,

$$\begin{aligned}
\log|\tilde{x}| &= \log\left(\sum_{j=0}^{n} \left| \frac{hGH_j}{H} \right|^2 \right)^{1/2} \\
&= \log|G| + \log\left(\sum_{j=0}^{n} |H_j|^2 \right)^{1/2} + \log|h| - \log|H| \\
&\le \log|\tilde{y}| + (n+1)\sum_{k=1}^{n+1} \log|\tilde{\alpha}^{(k)}| + \log|h| - \log|H| + \log B^n (n+1)^{(n+1)/2},
\end{aligned}$$

so, with a suitable constant K_2,

$$\begin{aligned}
T(r,x) \le\ & T(r,y) + (n+1)\sum_{k=1}^{n+1} T(r,\alpha^{(k)}) + N(r,0,h) \\
& - N(r,\infty,h) - N(r,0,H) + K_2.
\end{aligned}$$

Since $N(r,0,h) \le N(r,0,H)$, we obtain

$$T(r,x) \le T(r,y) + (n+1)\sum_{k=1}^{n+1} T(r,\alpha^{(k)}) + K_2. \tag{5.7}$$

Next (5.4) and (5.6) imply

$$N(r,0,G_k) \le \sum_{j=0,\,\neq k}^{n} T(r,\alpha^{(j)}) + K_3 \tag{5.8}$$

and

$$N(r,0,G) \le \sum_{j=0}^{n} T(r,\alpha^{(j)}) + K_4, \tag{5.9}$$

where K_3 and K_4 are constants. Hence, using (5.8) we have

$$\begin{aligned} N(r,0,y_{k-1}) &\le N(r,0,F_k) + N(r,0,G_k) \\ &\le N(r,0,F_k) + \sum_{j=0,\,\neq k}^{n} T(r,\alpha^{(j)}) + K_4. \end{aligned} \tag{5.10}$$

Since $|H| \le (\prod_{j=1}^{n+1} |G_j|)|G|$, we have

$$N(r,0,H) \le N(r,0,G) + \sum_{j=1}^{n+1} N(r,0,G_k) + K_5, \tag{5.11}$$

where K_5 is a constant. Hence, using $N(r,0,h) \le N(r,0,H)$, together with (5.8), (5.9), and (5.11), we have

$$\begin{aligned} N(r,0,F_k) &\le N(r,0,y_{k-1}) + N(r,0,G) + N(r,0,h) \\ &\le N(r,0,y_{k-1}) + 2N(r,0,G) + \sum_{j=1}^{n+1} N(r,0,G_j) + K_5 \\ &\le N(r,0,y_{k-1}) + (n+2)\sum_{j=1}^{n+1} T(r,\alpha^{(j)}) + K_6, \end{aligned} \tag{5.12}$$

where K_6 is a suitable constant. Therefore using (5.1) in (5.5), (5.7), (5.10), and (5.12), we obtain (5.2) and (5.3).

Now it follows from (5.2) and (5.3) that

$$\delta(\alpha^{(k)},x) = 1 - \limsup_{r\to\infty} \frac{N(r,0,y_{k-1})}{T(r,y)} = \delta(a^{(k)},y) \qquad (k = 1,2,\ldots,n+1),$$

where $a^{(k)} \in (P_nC)^*$ are induced by $(0,\ldots,0,1,0,\ldots,0)$ ($(k-1)$th spot). Therefore, Theorem 3 follows Theorem 2 in Toda [**14**].

6. Proof of Theorem 1(II). We may assume that $\delta(\alpha^{(k)}) \ge 1 - \cos\pi\mu$, $k = 1,2,\ldots,n$. Then

$$(4/\mu)\sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} \ge 2\pi,$$

so

$$\min\{2\pi, (4/\mu)\sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2}\} = 2\pi, \qquad k = 1,\ldots,n.$$

Hence, it follows from Lemma 5 that

$$\sum_{k=n+1}^{q} \min\left\{2\pi, \frac{4}{\mu}\sin^{-1}\left(\frac{\delta(\alpha^{(k)})}{2}\right)^{1/2}\right\} = 0$$

and, consequently,

$$\delta(\alpha^{(k)}) = 0, \qquad k = n+1, \ldots, q.$$

We next assume that $\delta(\alpha^{(k)}) > 1 - \cos \pi\mu$ ($k = 1, \ldots, p$; $0 \le p < n$) and $\delta(\alpha^{(k)}) \le 1 - \cos \pi\mu$ ($k = p+1, \ldots, q$). Then it follows from Lemma 5 that

$$\sum_{k=p+1}^{q} \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \le 2(n-p)\pi.$$

Hence, from the reasoning of §3 of Edrei [**5**] we can deduce that

$$\sum_{k=p+1}^{q} \delta(\alpha^{(k)}) \le (n-p)(1 - \cos \pi\mu). \tag{6.1}$$

It follows from Lemma 1 of Edrei [**5**] that equality in (6.1) is possible if and only if exactly $n - p$ of $\{\delta(\alpha^{(k)})\}_{k=p+1}^{q}$ are equal to $1 - \cos \pi\mu$ and all other $\delta(\alpha^{(k)})$ are equal to zero.

Thus the proof of Theorem 1(II) is complete.

7. Proof of Theorem 1(III). Our assumption $\mu > 1/2$ implies

$$(4/\mu) \sin^{-1}(\delta(\alpha^{(k)})/2)^{1/2} < 2\pi,$$

so it follows from Lemma 5 that

$$\sum_{k=1}^{q} \frac{4}{\mu} \sin^{-1} \left(\frac{\delta(\alpha^{(k)})}{2} \right)^{1/2} \le 2n\pi.$$

Hence, from the reasoning of §2 of Edrei [**5**] we can deduce that

$$\sum_{k=1}^{q} \delta(\alpha^{(k)}) \le [2n\mu] + 1 - \cos(2n\mu - [2n\mu])(\pi/2),$$

which proves Theorem 1(III).

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